Infinite Hidden Markov Models for Multiple Multivariate Time Series with Missing Data

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Low-Cost, Real-Time Sensors

- Air pollution exposure causes a wide variety of negative health outcomes
- Most studies rely in area-level exposure such as from a centrally located monitor or modeled exposure surface
- Low-cost, real-time sensors offer the promise to measure air pollution at the individual level
- Measure exposure high temporal resolution
- Moves with individuals

[1] Figure source: https://finance.yahoo.com

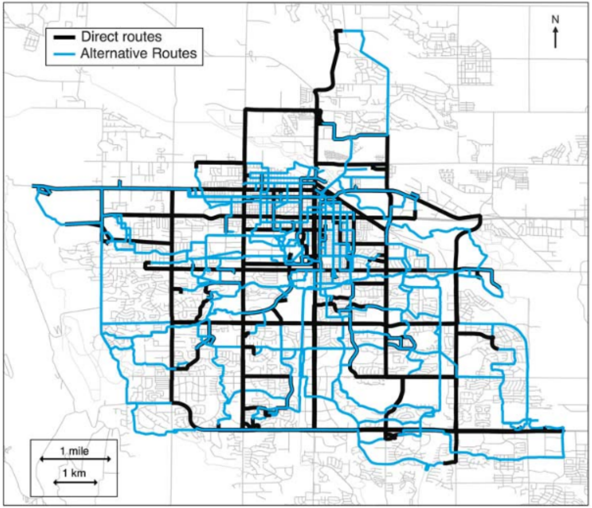
Low-Cost, Real-Time Sensors

- Lots of data
- Lots of promise
- Lots of challenges

[1]Figure source: https://finance.yahoo.com

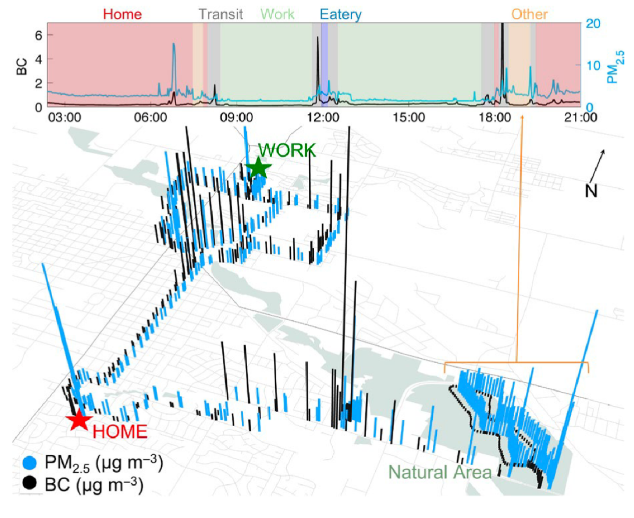
Fort Collins Commuter Study (FCCS)

- 45 individuals
- 1 to 13 non-consecutive days each
- Exposure measured for
  - black carbon (BC)
  - carbon monoxide (CO)
  - fine particulate matter ($PM_{2.5}$)
- Exposure at 10 second intervals



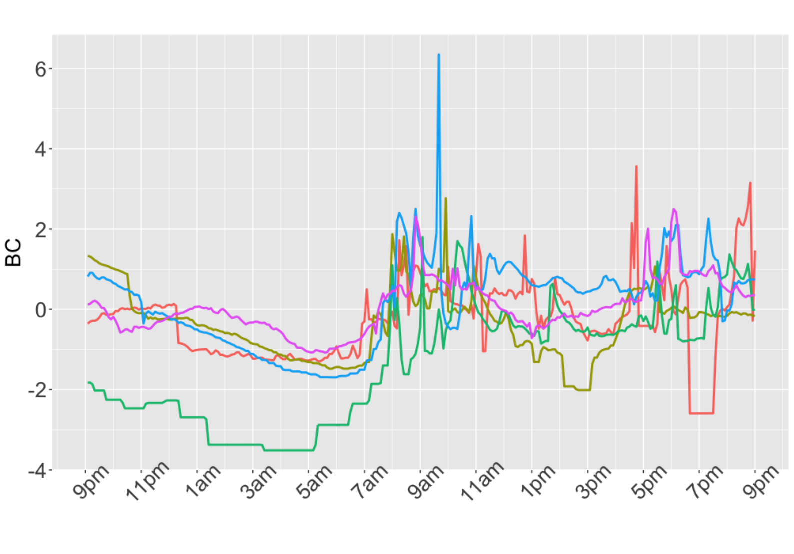
[1]Figure source: Good et al. (2016) *J. of Exposure Science & Environmental Epidemiology*.

Fort Collins Commuter Study (FCCS)



[1]Figure source: Koehler et al. (2019) *Indoor Air*.

Fort Collins Commuter Study (FCCS)



Statistical Challenges of Low-Cost, Real-Time Sensors

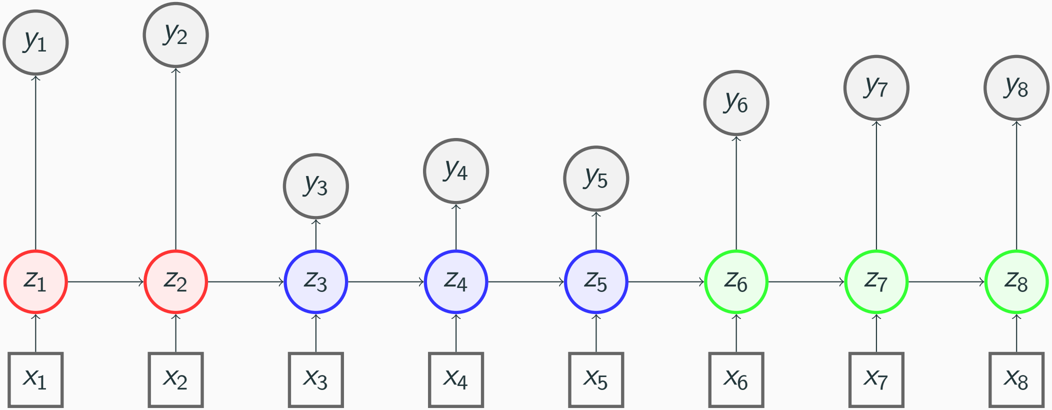
- Missing data due to
    - user non-compliance
    - device failure
    - levels below limit of detection
- How to classify exposure?
- How to relate to health outcomes?

Our Objectives

- Goals:
    - Find shared patterns in exposures
    - Impute missing data
- Want to do this in a way that allows for rapid changing of microenvironment and shared trends / common spaces
- Previous imputation methods either ignore temporal ordering of data or treat it as smoothly varying (Arroyo et al., 2018; Molitor et al., 2006; Krall et al., 2015; Houseman et al., 2017)

Hidden Markov Model

x = covariates, z = hidden states, y = exposure data



Infinite Hidden State Model: Notation

- Observed data
  - $\mathbf{y}_{ist}$ is the vector of exposures for individual i on day s at time t
  - $\mathbf{Y}_{is,1:T_{is}}$ is the full multivariate time series for individual i on sampling day s
  - $\mathbf{x}_{ist}$ is a set of covariates
    - time of day
    - individual characteristics
    - user reported activity or microenvironment (e.g. home, work, transit, etc.)
- Latent structure
  - z_{ist} is a categorical factor representing the latent state assignment
  - $z_{ist} = k$ if individual i is in latent state k at at time t on day s
  - iHMM allows for unknown number of hidden states

Infinite Hidden State Model: Key Assumptions

- Conditional independence of observed data conditional on the hidden states

$$f(\mathbf{y}_{it}|\mathbf{y}_{i,1:t-1}, \mathbf{z}_{i,1:t}) = f(\mathbf{y}_{it}|z_{it})$$

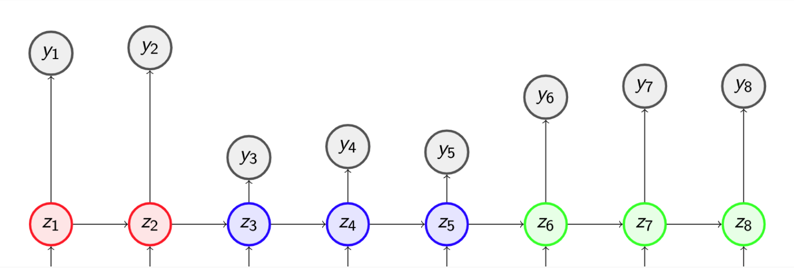
- Latent states follow the first-order Markov property

$$p(z_{it}|z_{i,1:t-1}) = p(z_{it}|z_{i,t-1})$$

Multivariate Normal Emission Distribution

Exposure data for individual i, sampling day s, and time point t is modeled

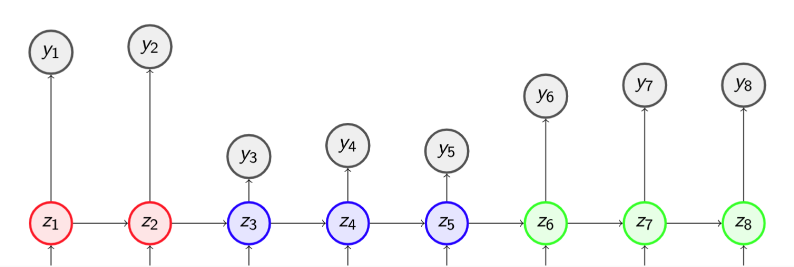
$$\begin{aligned} \mathbf{y}_{ist} | z_{ist} = k &\sim \mathrm{N}(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) \\ \boldsymbol{\mu}_k | \boldsymbol{\Sigma}_k &\sim \mathrm{N}\left(\mathbf{0}, \frac{1}{\lambda}\boldsymbol{\Sigma}_k\right) \\ \boldsymbol{\Sigma}_k &\sim \text{Inverse Wishart}\left(\nu, \mathbf{I}_p\right) \end{aligned}$$



Hidden State Model

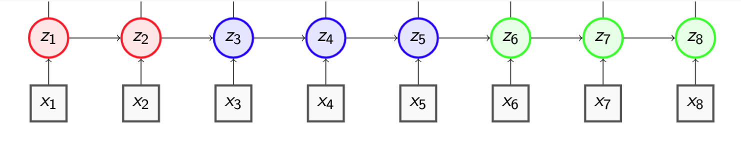
Model hidden states with hidden Markov model

$$z_{it} | z_{i,t-1} \sim \text{Categorical}\left(\pi_{z_{i,t-1}}\right)$$



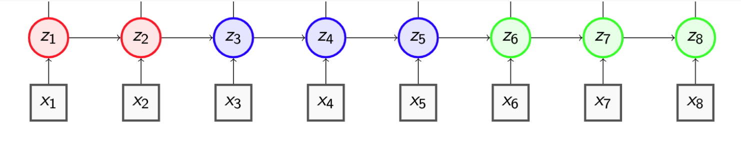
z = hidden states, y = exposure data

Probit Stick-Breaking Process on the Transition Distribution



- Want the state assignments and/or transitions to be covariate dependent
  - diurnal patterns
  - shared states among repeated sampling days for an individual
  - auxiliary information like time diary data
  - patterns in transitions

Probit Stick-Breaking Process on the Transition Distribution



- The probability of individual i on sampling day s transitioning from state j to state k at time t is

$$\begin{aligned}\pi_{jk}(\mathbf{x}_{ist}) &\equiv P(z_{ist} = k | z_{is,t-1} = j, \mathbf{x}_{ist}) \\ &= \Phi(\alpha_{jk} + \mathbf{x}'_{ist}\beta_k + \mathbf{x}'_{ist}\gamma_{ik}) \prod_{l<k} \{1 - \Phi(\alpha_{jl} + \mathbf{x}'_{ist}\beta_l + \mathbf{x}'_{ist}\gamma_{il})\}\end{aligned}$$

- α_{jk} controls state transitions at consecutive time points
- β_k and γ_{ik} control covariate-dependent and subject-specific trends

Prior Distributions on Transition Parameters

Transitions among states

$$\begin{aligned} \alpha_{jk} | \sigma_\alpha^2 &\sim \mathrm{N}(0, \sigma_\alpha^2) \text{ for } j \neq k \\ \sigma_\alpha^{-2} &\sim \mathrm{Gamma}(1, 1) \end{aligned}$$

Self-transitions

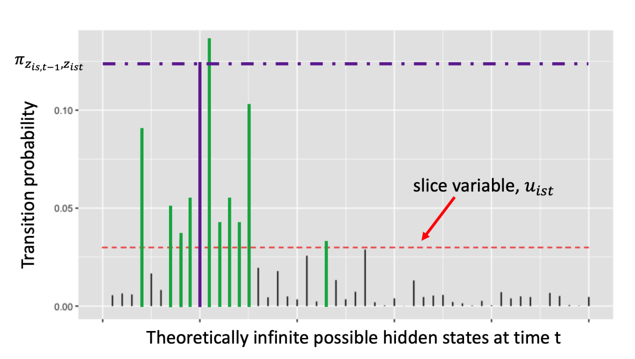
$$\begin{aligned} \alpha_{jj} | m_\alpha, v_\alpha &\sim \mathrm{N}(m_\alpha, v_\alpha) \\ m_\alpha &\sim \mathrm{N}(0, 1) \\ v_\alpha^{-1} &\sim \mathrm{Gamma}(1, 1) \end{aligned}$$

Covariate effects

$$\begin{aligned} \boldsymbol{\beta}_k &\sim \mathrm{N}(\mathbf{0}, \mathbf{I}) \\ \boldsymbol{\gamma}_{ik} | \kappa^2 &\sim \mathrm{N}(\mathbf{0}, \kappa^2 \mathbf{I}) \\ \kappa^{-2} &\sim \mathrm{Gamma}(1, 1) \end{aligned}$$

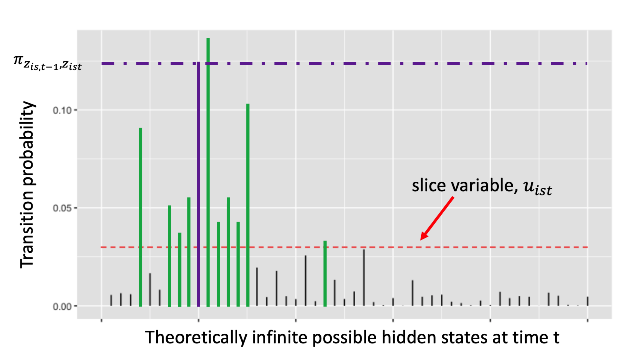
Posterior Sampling

- "Beam" sampling = slice sampling + dynamic programming (Van Gael et al., 2008)
- Sample entire hidden state trajectories at once
- Better mixing and faster convergence than Gibbs sampling



Posterior Sampling

- Slice sampling reduces problem to finite number of paths
- Forward pass calculates probabilities of each path
- Backwards step samples latent sequence



Imputation Model

- Impute missing data conditional on the state assignment using multivariate normal conditionals
- MAR for all components

$$\mathbf{y}_{it,\text{MAR}} | z_{it} = k, \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k \sim \text{N}\left(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k\right)$$

- MAR for only some components use conditional distributions of

$$\begin{bmatrix} \mathbf{y}_{it,\text{obs}} \\ \mathbf{y}_{it,\text{MAR}} \end{bmatrix} \Bigg| z_{it} = k, \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k \sim \text{N}\left(\begin{bmatrix} \boldsymbol{\mu}_{(k,\text{obs})} \\ \boldsymbol{\mu}_{(k,\text{MAR})} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{(k,\text{obs},\text{obs})} & \boldsymbol{\Sigma}_{(k,\text{obs},\text{MAR})} \\ \boldsymbol{\Sigma}_{(k,\text{MAR},\text{obs})} & \boldsymbol{\Sigma}_{(k,\text{MAR},\text{MAR})} \end{bmatrix} \right)$$

- For data missing below LOD similar but from a truncated multivariate distribution
  - We know the LOD and know if data is below LOD or MAR

Simulation Study

Purpose

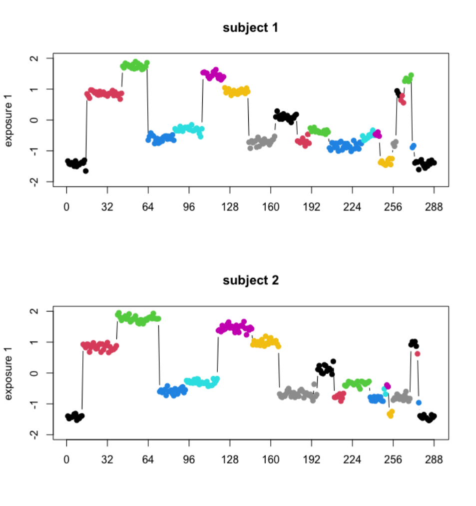
- Evaluate parameter estimation
- Evaluate imputations
- Compare with competing methods

Simulation Scenarios

- Shared cyclical trends
- Distinct cyclical trends

Missing Data Levels

- 0%, 5%, 10%, 20%
- MAR and below LOD

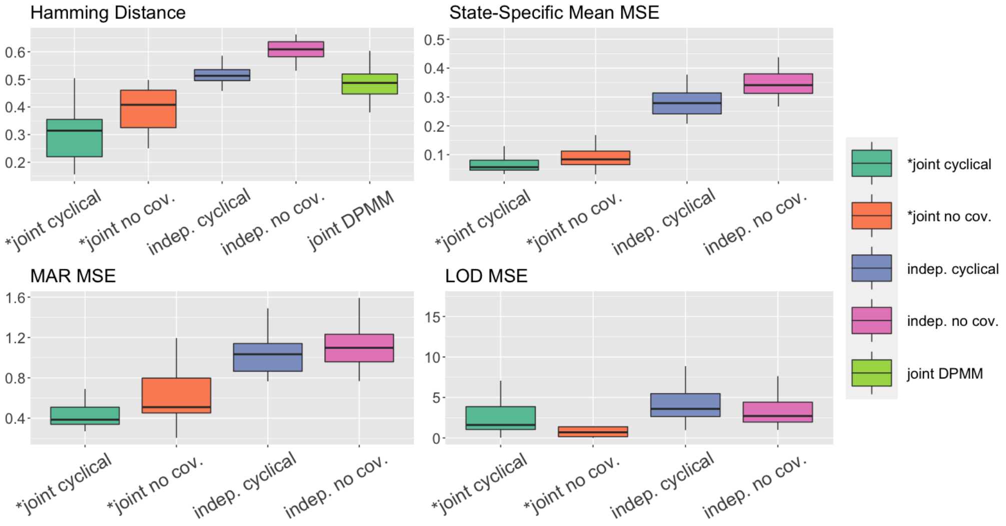


Models for Simulation Study

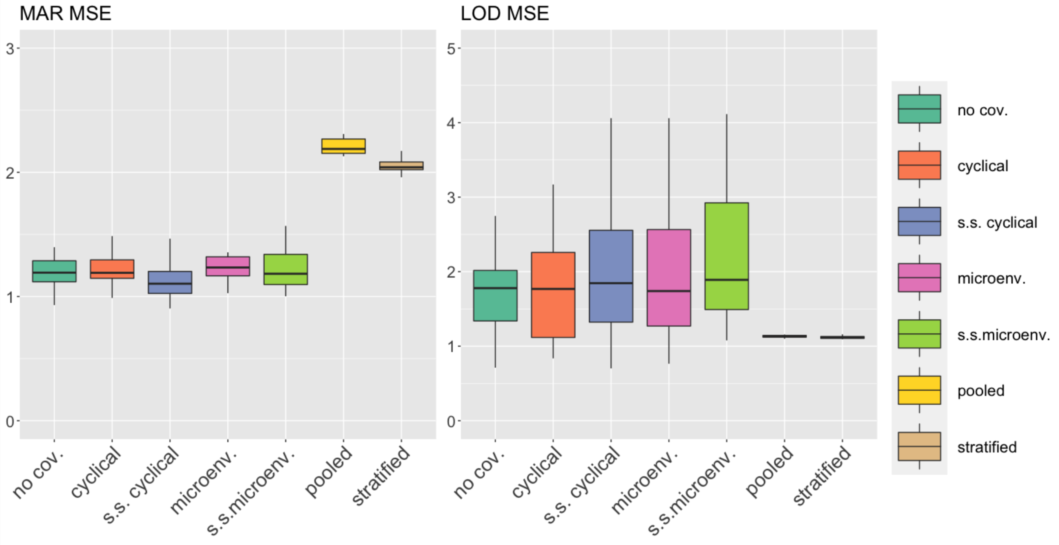
- ***Joint no covariates**: shared states, no covariates
- ***Joint cyclical**: shared states and shared cyclical trends
- **Independent no covariates**: individual states, no covariates
- **Independent cyclical**: individual states and individual cyclical trends
- **Joint DPMM**: Dirichlet process mixture model, shared states, no temporal dependency

*proposed methods

Shared Scenario Results 5% Missing Data



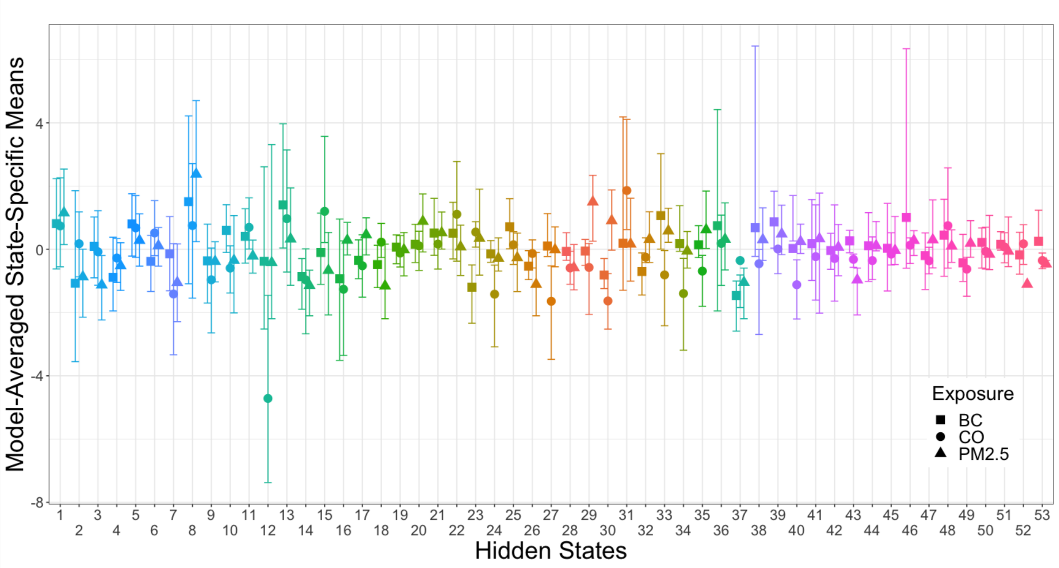
Validation Study Results



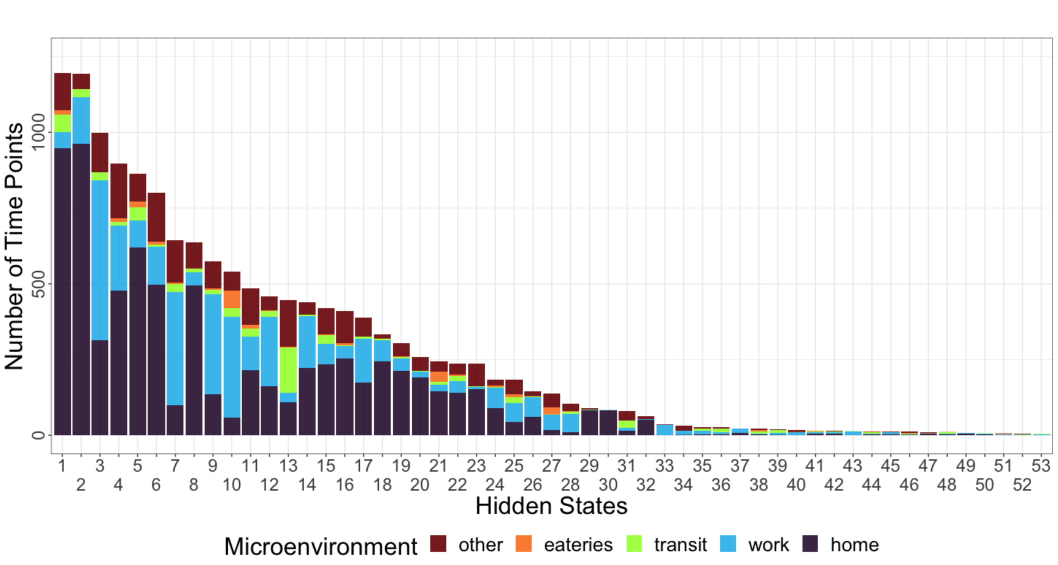
Fort Collins Commuter Study (FCCS)

- Fit joint model with cyclical trends to the data
- Considered average exposure over five minute intervals
- Trimmed data to be 24 hour segments

State-Specific Mean Estimation



Hidden State Correspondence with Microenvironments



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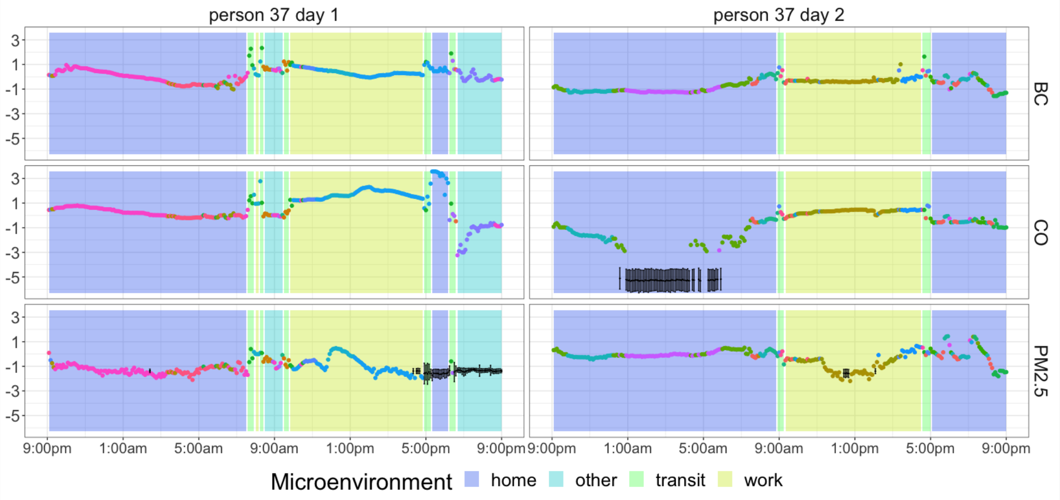

Summary

- Developed statistical method for analyzing multiple multivariate time series with missing data
- PSBP on transition distribution to estimate an unknown number of hidden states
- Multiple imputation for data that are MAR and below the LOD
- Demonstrated our method's estimation and imputation gains over competing approaches in simulation and validation studies
- Applied method to FCCS data to impute missing exposure data and identify time-activity patterns associated with exposures
- Many more challenges with low-cost, real-time sensor data

Thank You

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